The structure of a shock wave in a fully ionized gas

By J. D. JUKES
Atomic Energy Research Establishment, Harwell, Berkshire

(*Received* 2 *August* 1957)

SUMMARY

The structure of a plane shock wave moving through a completely ionized plasma of protons and electrons is calculated. It is assumed that the two species of particles behave as two gases, each separately in a quasi-equilibrium state corresponding generally to two different temperatures. Navier–Stokes type equations with coefficients of viscosity and thermal conductivity appropriate to the two species are solved by numerical iteration.

For very strong shocks it is found that both the velocity of electrons and protons and the temperature of the protons change in a distance about twice the mean path for momentum transfer between protons in the hot (shocked) gas. The electron temperature changes in about eight of these mean free paths, causing a relatively wide zone of hot electrons at low density ahead of the usual velocity shock-front. The density and temperature gradients of protons and electrons create an electric field.

1. INTRODUCTION

A study of the structure of a shock wave requires consideration of the dissipative effects of viscosity and thermal conductivity. Suitable macroscopic equations can be derived from kinetic theory once the Boltzmann equation is solved for the velocity distribution function. Generally this function can only be found by a method of successive approximation with the Maxwell (equilibrium) distribution as the zero order term (see Chapman & Cowling 1939). To ensure convergence of this solution it is necessary for the ratio of the mean free path for momentum transfer in the fluid to be small compared with some characteristic length in which the distribution function changes appreciably. This is equivalent to an assumption of quasi-equilibrium conditions. Zero order terms give equations for a frictionless fluid and the result of a vanishingly thin shock front. First-order terms yield the familiar Navier–Stokes [N–S] equations, containing the effects of viscosity and thermal conductivity in two coefficients, μ and λ.

Becker (1923) solved the N–S equations assuming μ and λ to be constant. Later, Thomas (1944) included a temperature variation in μ and λ appropriate to a model gas of hard-sphere elastic molecules. This inclusion is important for strong shocks where there are large temperature and density changes because these variations lead to large changes in μ and λ. With this inclusion, it is found that the shock front thickness tends to a finite limit as the Mach number M tends to infinity; a result not obtained with the simple Becker solution.

Mott-Smith (1951) has criticized the application of the N–S equations to strong shock waves ($M > 2$), where both experiments and more refined theory suggest that physical quantities and the distribution function change appreciably in distances only once or twice the mean free path for momentum transfer between molecules. It is unlikely that the expansion solution of Chapman & Cowling can be valid in these circumstances. Mott-Smith, although still only dealing with a gas of molecules of one type, assumes a distribution function composed of two Maxwellian distributions corresponding to upstream and downstream temperatures. Using these, he solves a transport equation. His theory predicts shocks whose limiting thickness as $M \to \infty$ is rather more than that of Thomas's theory.

The present paper develops the N–S equations for a fully ionized gas, such as may be encountered in some electrical gas-discharge. The treatment is similar to that of Thomas with the important difference that such a plasma is a mixture of two species of charged particles. Each species is assumed to behave as a gas in a quasi-equilibrium state. There is a separate, approximately Maxwellian, distribution for each gas, corresponding in general with two different temperatures for the proton and electron gases.

In the light of Mott-Smith's results this treatment should follow Thomas in arriving at a lower limit for the shock thickness, at the same time yielding the essential qualitative features of the shock front structure.

2. Plasma properties

The calculations are restricted to a plasma of protons and electrons in which simple equations of state are valid (Spitzer 1956). Any externally applied electric or magnetic fields are neglected. The equations of state are then,

$$p_e = n_e k T_e, \qquad p_i = n_i k T_i, \tag{2.1 a}$$

where k is Boltzmann's constant, p, n and T are the partial pressures, the number densities and the temperatures of electrons and protons as indicated by the suffices (i is for the heavy ion, a proton in this case). For a gas of single particles with no internal degrees of freedom the specific heat ratio, $c_p/c_v \equiv \gamma$, is 5/3. In the hydrodynamic equations it will be supposed that the electrostatic force acting on the plasma is small compared to the fluid forces arising from the pressure and momentum. This assumption will be justified later when the electric field has been estimated. The assumption implies that the charge separation is negligible, so that to good approximation,

$$n_i \doteqdot n_e = n. \tag{2.1 b}$$

Because protons are nearly two thousand times heavier than electrons, the protons are chiefly responsible for the transport of momentum (viscosity) while the electrons are chiefly responsible for the transport of random, or thermal, energy. It is therefore reasonable to take the viscosity to be a function of proton temperature alone and the thermal conductivity a function of electron temperature alone, provided

$$\lambda_e \frac{dT_e}{dx} \gg \lambda_i \frac{dT_i}{dx}$$

and

$$\mu_i \frac{du_i}{dx} \gg \mu_e \frac{du_e}{dx},$$

where λ_e denotes the thermal conductivity due to electrons alone, and so forth, and u_j is the streaming velocity of each species. The results are found to satisfy these conditions except possibly for very strong shocks ($M > 10$) where the second condition may not hold at the beginning of the shock. But the electron viscosity is there much smaller than the average viscosity throughout the shock so the qualitative results should not be invalid. The dependence of the coefficients μ and λ on the temperatures under the above conditions has been found by Chapman & Cowling and others.

As in other problems of dissipation in gases it is convenient to compute the Prandtl number $P \equiv \mu C_p/\lambda$ of a plasma under equilibrium conditions. An exact treatment by Chapman & Cowling for a proton–electron plasma yields $P = 0{\cdot}065$. This is much less than is found for a one-particle gas ($\sim 0{\cdot}6$) because the mobility of electrons far exceeds that of protons by a factor of order $(m_i/m_e)^{1/2} \doteqdot 43$ when electron and proton temperatures are equal.

There is a slow interchange of energy between protons and electrons at different temperatures. Post (1956) gives the rate of energy transfer in collisions from a single proton to an electron gas at temperature T_e. By integrating over a Maxwell distribution of protons of temperature T_i, the rate of heat transfer per unit volume between the two gases is found to be

$$\left(\frac{32\pi}{m_e k}\right)^{1/2} e^4 \log_e \Lambda \frac{m_e n^2}{m_i T_e^{3/2}} |T_e - T_i|, \tag{2.2}$$

where $\log_e \Lambda$ is the Coulomb logarithm of Spitzer's plasma theory (1956).

3. Equations

Suppose the plasma flows in planes parallel to the x-axis from $-\infty$ to $+\infty$ and that conditions of thermal equilibrium exist at upstream and downstream infinity, denoted by the points (1) and (2), with the usual Rankine–Hugoniot relations between them. Assume a steady state shock wave exists somewhere in the flow and consider each species of particle separately. For the jth species, the rate at which heat is supplied to an elemental unit volume moving with the flow is given by the mobile derivative,

$$\frac{DQ_j}{Dt} = ku_j\left(\frac{3}{2} n_j \frac{dT_j}{dx} - T_j \frac{dn_j}{dx}\right), \tag{3.1}$$

where $u_j = u$ = fluid velocity, and $n_j = n$ = number density. Carrying out the transformation from the moving axes to a system of axes fixed relative to the shock wave, the equations of energy transfer become:

$$ku\left(\frac{3}{2}n\frac{dT_i}{dx} - T_i\frac{dn}{dx}\right) = \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 + \frac{Kn^2}{T_e^{3/2}}(T_e - T_i), \tag{3.2}$$

$$ku\left(\frac{3}{2}n\frac{dT_e}{dx} - T_e\frac{dn}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dT_e}{dx}\right) - \frac{Kn^2}{T_e^{3/2}}(T_e - T_i). \tag{3.3}$$

K is a function of the atomic constants, and is found from the heat transfer term (2.2) to be $(32\pi/m_e k)^{1/2}e^4(m_e/m_i)\log\Lambda$.

The momentum equation, neglecting electrostatic forces, is

$$m\frac{du}{dx} = -\frac{dp_t}{dx} + \frac{d}{dx}\left(\frac{4}{3}\mu\frac{du}{dx}\right), \tag{3.4}$$

where $p_t \equiv p_e + p_i$ is the total pressure and

$$m = \rho_t u \tag{3.5}$$

is the mass flow per unit area. The continuity relation is

$$dm/dx = 0. \tag{3.6}$$

The integration of (3.4) and elimination of the constant of integration by means of the conditions at $x = \pm\infty$ gives

$$p_t + \rho_t u^2 - \frac{4}{3}\mu\frac{du}{dx} = (p_t + \rho_t u^2)_1 = (p_t + \rho_t u^2)_2. \tag{3.7}$$

(3.2), (3.3), (3.6) and (3.7) are the fundamental equations. We have, also,

$$\rho_t = n(m_i + m_e), \qquad p_t = nk(T_i + T_e),$$

and from (3.5) and (3.6)

$$\rho_t u = \text{constant} \doteqdot num_i = n_1 u_1 m_i, \tag{3.8}$$

and therefore

$$\frac{du}{dx} = -\frac{n_1 u_1}{n^2}\frac{dn}{dx}. \tag{3.9}$$

Add (3.2) and (3.3) and substitute $\mu(du/dx)$ from (3.7) to get the total energy equation

$$ku\left[\frac{3}{2}n\frac{d}{dx}(T_e + T_i) - (T_e + T_i)\frac{dn}{dx}\right]$$
$$= [p_t + \rho_t u^2 - (p_t + \rho_t u^2)_1]\frac{du}{dx} + \frac{d}{dx}\left(\lambda\frac{dT_e}{dx}\right). \tag{3.10}$$

Substitute from (3.8) and (3.9) and then integrate (3.10) with respect to x to give

$$\frac{3}{2}u_1 n_1 k(T_e + T_i) = -\frac{2n_1^2 kT_1 u_1}{n} - \frac{n_1^2 m_i u_1^3}{n} + \frac{m_i u_1^3 n_1^3}{2n^2} + \lambda\frac{dT_e}{dx} + \text{const.} \tag{3.11}$$

At $x = -\infty$, $T_i = T_e = T_1$, and the elimination of the constant gives

$$\frac{3}{2}u_1 n_1 k(T_e + T_i) + \frac{2n_1^2 kT_1 u_1}{n} + \frac{n_1^2 m_i u_1^3}{n} -$$
$$- \frac{m_i u_1^3 n_1^3}{2n^2} - \lambda\frac{dT_e}{dx} = 5u_1 n_1 kT_1 + \frac{n_1 m_i u_1^3}{2}. \tag{3.12}$$

On substituting from (3.7) to (3.9), equations (3.3) and (3.7) become

$$n_1 u_1 k\left(\frac{3}{2}\frac{dT_e}{dx} - \frac{T_e}{n}\frac{dn}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dT_e}{dx}\right) - \frac{Kn^2}{T_e^{3/2}}(T_e - T_i), \tag{3.13}$$

and

$$\frac{4}{3}\mu\frac{n_1 u_1}{n^2}\left(\frac{dn}{dx}\right) = 2n_1 kT_1 + n_1 m_i u_1^2 - nk(T_e + T_i) - \frac{n_1^2 m_i u_1^2}{n}. \tag{3.14}$$

From now on we work with (3.12), (3.13) and (3.14), making them dimensionless by the substitutions

$\nu \equiv n/n_1$, $\tau \equiv T_e/T_1$, $\theta \equiv T_i/T_1$, $P \equiv \mu_1 c_p/\lambda_1$, $\pi \equiv p_t/p_{t1}$, $\xi \equiv x/l$,

where l = mean free path of simple kinetic theory for ions ahead of the shock so that $l \equiv 4\mu_1/(3n_1 m_i c_1)$, c_1 being the velocity of sound ahead of the shock.

The Mach number of the shock front is

$$M^2 \equiv \left(\frac{u_1}{c_1}\right)^2 = \frac{m_i u_1^2}{2\gamma kT_1}.$$

From (3.12), (3.13) and (3.14)

$$\frac{3}{2}(\theta+\tau) + \frac{2}{\nu} + \frac{2\gamma}{\nu}M^2 - \frac{\gamma M^2}{\nu^2} - \frac{\lambda}{\lambda_1}\left(\frac{3}{2}\frac{\gamma}{\gamma-1}\frac{1}{PM}\right)\frac{d\tau}{d\xi} = 5 + \gamma M^2, \tag{3.15}$$

$$\frac{\theta^{5/2}}{\nu^2}\frac{d\nu}{d\xi} = \frac{1}{\gamma M} - \frac{\nu}{2\gamma}\left(\frac{\tau+\theta}{M}\right) + M\left(1 - \frac{1}{\nu}\right), \tag{3.16}$$

$$\frac{3}{2}\frac{d\tau}{d\xi} - \frac{\tau}{\nu}\frac{d\nu}{d\xi} = \frac{3}{2}\frac{\gamma/(\gamma-1)}{PM}\frac{d}{d\xi}\left(\frac{\lambda}{\lambda_1}\frac{d\tau}{d\xi}\right) - \frac{EP}{M}\frac{\nu^2(\tau-\theta)}{\tau^{3/2}}. \tag{3.17}$$

E is a pure number, which, in terms of K, is

$$\frac{(\gamma-1)m_i\lambda_1 K}{3\gamma^2 k^{1/2}(kT_1)^{5/2}}.$$

Taking γ as 5/3, and using the value of λ_1 for zero current given by Spitzer, E is found to be 0·77.

Chapman & Cowling give

$$\mu/\mu_1 = \theta^{5/2} \quad \text{and} \quad \lambda/\lambda_1 = \tau^{5/2},$$

so that (3.15), (3.16) and (3.17) become

$$\frac{3}{2}(\theta+\tau) + \frac{2}{\nu} - \frac{10}{3}\frac{M^2}{\nu} - \frac{5}{3}\frac{M^2}{\nu^2} - \frac{15}{4}\frac{\tau^{5/2}}{PM}\frac{d\tau}{d\xi} = 5 + \frac{5}{3}M^2, \tag{3.18}$$

$$\frac{\theta^{5/2}}{\nu^2}\frac{d\nu}{d\xi} = \frac{3}{5M} - \frac{3\nu(\tau+\theta)}{10M} + M\left(1 - \frac{1}{\nu}\right), \tag{3.19}$$

$$\frac{3}{2}\frac{d\tau}{d\xi} - \frac{\tau}{\nu}\frac{d\nu}{d\xi} = \frac{15}{4PM}\frac{d}{d\xi}\left(\tau^{5/2}\frac{d\tau}{d\xi}\right) - \frac{EP}{M}\frac{\nu^2}{\tau^{3/2}}(\tau-\theta). \tag{3.20}$$

These equations reduce to,

$$N \equiv M\frac{d\nu}{d\xi} = \frac{\nu^2}{\theta^{5/2}}\left[M^2\left(\frac{\nu-1}{\nu}\right) + \frac{3}{5}\left\{1 - \frac{\nu(\tau+\theta)}{2}\right\}\right], \tag{3.21}$$

$$T \equiv M\frac{d\tau}{d\xi} = \frac{2PM^2}{5\tau^{5/2}}\left[\tau + \theta + \frac{4}{3\nu} - \frac{10}{3} - \frac{10}{9}M^2\left(\frac{\nu-1}{\nu}\right)^2\right], \tag{3.22}$$

$$H \equiv M\frac{d\theta}{d\xi} = \frac{2}{3}EP\frac{\nu^2}{\tau^{3/2}}(\tau-\theta) + \\ + \frac{1}{\theta^{5/2}}\left[\frac{20}{9}M^2\left(\frac{\nu-1}{\nu}\right)+\frac{4}{3}-\frac{2}{3}\tau\nu\right]\left[M^2\left(\frac{\nu-1}{\nu}\right)+\frac{3}{5}\left\{1-\frac{\nu(\tau+\theta)}{2}\right\}\right] \\ = \frac{2}{3}EP\frac{\nu^2}{\tau^{3/2}}(\tau-\theta)+\frac{N}{\nu^2}\left[\frac{20}{9}M^2\left(\frac{\nu-1}{\nu}\right)+\frac{4}{3}-\frac{2}{3}\tau\nu\right]. \tag{3.23}$$

4. Solutions

The integral curves of equations (3.21)–(3.23) lie in a τ, ν, θ-space as ξ goes from $-\infty$ to $+\infty$. We draw the surfaces N, H and $T = 0$ and examine the solutions in planes corresponding to fixed values of one of the variables (e.g. figure 1 shows the $\nu-\theta$ plane with $\tau = 1$). In the 3-space

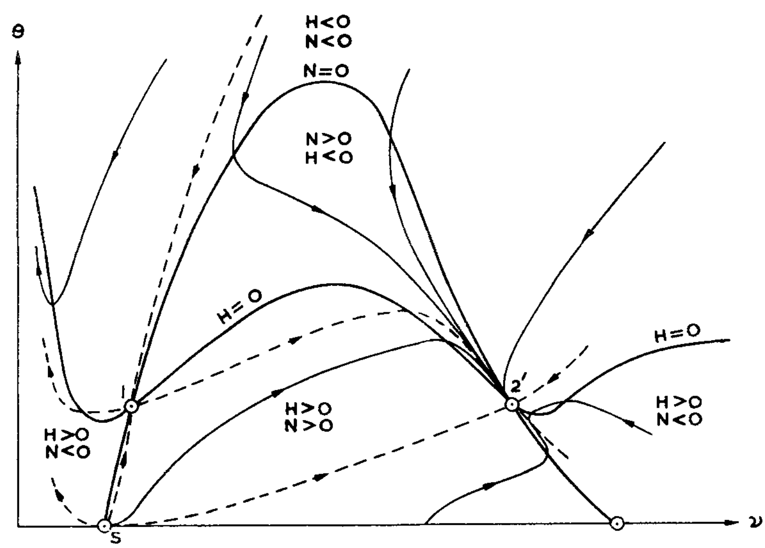

Figure 1. $\theta-\nu$ curves in $\tau = 1$ plane (not to scale). Arrows indicate ξ increasing. Strong shock, $M = 10$.

the curves pass through two saddle points (1) and (2) where $\xi = \mp\infty$, these being the intersections of the surfaces N, H, $T = 0$ by virtue of the boundary conditions. Singular points exist at $\tau = 0 = \theta$. Saddle point (1) contains the intersection of the planes ν, τ, $\theta = 1$ and saddle point (2) the planes ν, τ, $\theta = \nu_2$, τ_2, θ_2 respectively. In a 2-space, such as the plane shown in figure 1, the point of intersection of the plane $\tau = 1$ with the surfaces $N = 0 = H$ has been denoted by (2′) to distinguish it from the corresponding 3-space point (2). Similarly (2″) corresponds to (2) in figure 2.

It is necessary to investigate the behaviour of the solutions close to the saddle points and we do this by a linearization. There are two distinct sets of solutions for the extreme physical cases, (*a*) $M \gg 1$—a very strong shock, and (*b*) $M \gtrsim 1$—a weak shock, approximating to an acoustic wave.

Equations (3.15)–(3.17) may be written

$$a\frac{d^2\tau}{d\xi^2}+b\frac{d\tau}{d\xi}+c\frac{d\nu}{d\xi}=F(\tau,\nu),\qquad d\frac{d\tau}{d\xi}+f\frac{d\nu}{d\xi}=G(\nu),\tag{4.1}$$

where

$$\left.\begin{aligned}
&a\equiv\frac{15\tau^{3/2}}{4P^2E\nu^2},\qquad b\equiv\frac{5}{2}\frac{\tau^{5/2}}{PM}-\frac{3M}{2EP}\frac{\tau^{3/2}}{\nu^2}\\
&c\equiv\frac{M\tau^{5/2}}{EP\nu^3},\qquad d\equiv\frac{3\tau^{5/2}\nu}{4PM^2},\qquad f\equiv\frac{\theta^{5/2}}{\nu^2},\\
&F(\tau,\nu)\equiv 2\tau-\frac{10}{3}+\frac{4}{3\nu}-\frac{10M^2}{9}\left[1-\frac{2}{\nu}+\frac{1}{2\nu^2}\right],\\
&G(\nu)\equiv\frac{M}{3}\left(5-\nu-\frac{4}{\nu}\right)+\frac{(1-\nu)}{M}.
\end{aligned}\right\}\tag{4.2}$$

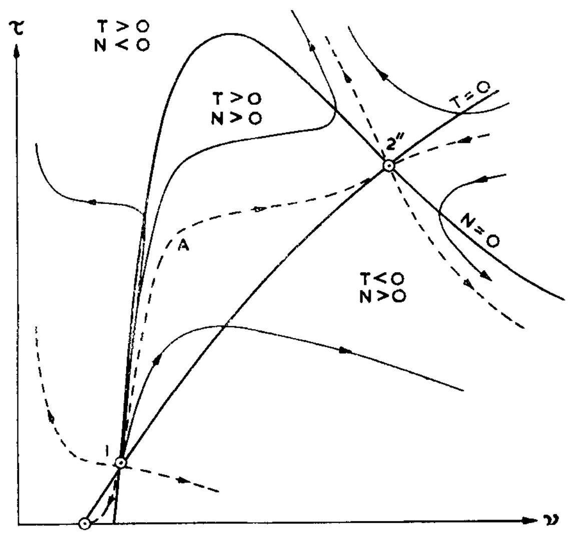

Figure 2. $\tau - \nu$ curves in $\theta = 1$ plane (not to scale). Arrows indicate ξ increasing. Strong shock, $M = 10$.

We linearize at points $i(=1, 2)$ by substituting

$$\left.\begin{aligned}\tau&=\tau_i+\Delta\tau\\ \nu&=\nu_i+\Delta\nu\\ \theta&=\theta_i+\Delta\theta\end{aligned}\right\},\qquad \text{where }\left.\begin{aligned}\Delta\tau&=H\exp\alpha\xi\\ \Delta\nu&=K\exp\alpha\xi\\ \Delta\theta&=L\exp\alpha\xi\end{aligned}\right\},\tag{4.3}$$

in (4.1), obtaining a cubic equation in α

$$(af)\alpha^3+(bf-aG_\nu-cd)\alpha^2+(F_\nu d-fF_\tau-bG_\nu)\alpha+F_\tau G_\nu=0,\tag{4.4}$$

where $G_\nu\equiv\partial G/\partial\nu$, etc. Also, from (4.1), $\Delta\tau/\Delta\nu=(G_\nu-\alpha f)/(\alpha d)$ and

$$\frac{\Delta\theta}{\Delta\nu}=\left(\frac{5\alpha\tau^{5/2}}{2PM}-1\right)\frac{\Delta\tau}{\Delta\nu}+\frac{4}{3\nu^2}+\frac{20M^2}{9}\frac{(\nu-1)}{\nu^3}.\tag{4.5}$$

Strong shocks $M \gg 1$

We make use of the approximations $M^2\gg 1$, $\tau_2 \doteqdot (5/16)M^2$, $\nu_2 \doteqdot 4$, to find three (real) roots of α at points (1) and (2).

At (1),

$$\left.\begin{aligned} \alpha_{11} &= \frac{2}{5}PM + \frac{2}{3}\frac{EP}{M} + \dots \text{ higher powers of } M^{-1} \\ \alpha_{12} &= M - \frac{4}{5M} + \dots \end{aligned}\right\}, \quad \text{as } \xi = -\infty,$$

$$\alpha_{13} = -\frac{4EP}{3M} + \dots, \qquad \text{as } \xi = +\infty. \tag{4.6}$$

At (2),

$$\left.\begin{aligned} \alpha_{21} &= -\frac{3{\cdot}1}{M^4} + \dots \\ \alpha_{22} &= -\frac{47}{M^4} + \dots \end{aligned}\right\}, \quad \text{as } \xi = +\infty,$$

$$\alpha_{23} = \frac{1{\cdot}5}{M^4} + \dots, \qquad \text{as } \xi = -\infty.$$

A few typical integral curves are drawn on figures 1 and 2. The limiting integral curves which pass through both saddle points are shown by broken lines. To represent a physical solution satisfying the boundary conditions an integral curve must pass through both points (1) and (2). Also, ν, τ and θ must remain real, positive, non-zero and finite all along the curve $-\infty < \xi < +\infty$.

Reference to figures 1 and 2 shows that only one curve can be a physical solution. Also it must leave (1) with $\Delta\tau/\Delta\nu$ positive and arrive at (2) with $\Delta\theta/\Delta\nu$ negative, the sense of direction deriving from the increase of ξ. The values of $\Delta\tau/\Delta\nu$ and $\Delta\theta/\Delta\nu$ are given in (4.5) in terms of α. On substituting the values of α from (4.6) it is found that the physical solution leaves (1) with a slope corresponding to α_{11} and arrives at (2) with a slope corresponding to α_{21}.

We may note that even in the acceptable solution entropy does not increase monotonically between states (1) and (2). Of course the *net charge* of entropy is always positive between (1) and (2).

Weak shocks $M \gtrsim 1$

The analysis is similar to that for strong shocks, however, in this case we put $M = 1+m$ and expand in power series in m, assuming m to be small.

We find two roots of (4.4) at (1) and (2)

$$\alpha_{11} \doteqdot \frac{4mP}{2P+1}, \qquad \text{as } \xi = -\infty,$$

$$\alpha_{21} \doteqdot -\frac{4mP}{2P+1}, \quad \text{as } \xi = +\infty.$$

In each case $\Delta\nu/\Delta\tau \doteqdot \frac{3}{2}$, $\Delta\theta/\Delta\tau \doteqdot 1$.

An analysis, as for strong shocks, shows that other roots of α give physically impossible solutions. An analytic solution, obtainable to this first order of approximation, shows that $\tau \doteqdot \theta$ throughout the wave.

Numerical integration for strong shocks

A qualitative appraisal of the physical solution for a strong shock is of great help in the numerical integration between points (1) and (2). We notice from (3.21) and (3.22) that $d\tau/d\nu = T/N \sim P \ll 1$, unless N is small too, and we trace the required integral curve in figure 2. Leaving (1) the integral curve runs very close to $H = 0 = N$ in the positive domain until it almost reaches the plane $\tau = \tau_2$. From some point A (say) the curve bends sharply away to approach (2). That the integral curve must proceed from A to (2) in a $\theta - \nu$ plane is clear from figure 1, for A lies in a domain where $N > 0$ and $\nu > 1$ close to the point (1).

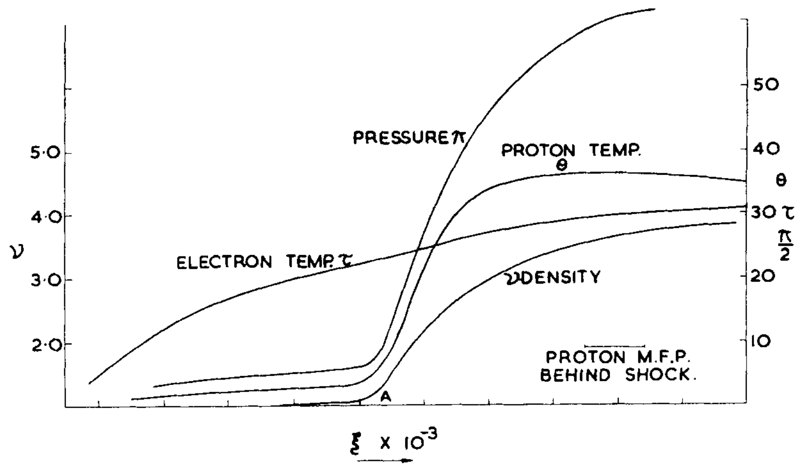

Figure 3. Shock profile. Strong shock, $M = 10$.

It is most important to note that the integral curves converge on point (2) in the $\theta - \nu$ plane, but on point (1) in the $\tau - \nu$ plane. To obtain a convergent solution by iteration one must take the following steps:

(i) Assume $\tau \doteqdot \tau_2$ with θ and $\nu \doteqdot 1$ between points (1) and A. Find ν and θ at A.

(ii) Proceed from A to (2) solving for ν and θ numerically.

(iii) Recalculate τ moving back from (2) to (1) using the first approximation to ν and θ. Calculate a second approximation to τ.

(iv) Recalculate ν and θ starting at (1), through A to (2). The iteration converges rapidly for $\tau_2 \gg 1$ (or $M \gg 1$).

(v) Integrate to obtain ν, θ, τ as function of ξ.

A solution has been obtained for a typical, very strong shock of Mach number 10. The results are shown in figure 3.

5. Strong shocks

Certain features of the structure of strong shocks may easily be deduced.

The hot electron zone

In a zone between points (1) and A the electron temperature is much higher than the proton temperature. To estimate the profile of T_e we assume $\theta \doteqdot \nu \doteqdot 1$ and substitute in (3.23), which gives

$$\frac{d\tau}{d\xi} = \frac{2}{5} PM\left(\frac{\tau - 1}{\tau^{5/2}}\right). \tag{5.1}$$

This can be integrated directly and the solution made continuous at A. (5.1) has the correct asymptotic behaviour as $\xi \to -\infty$. The zone has a characteristic thickness L, given by

$$L \sim l_2/2P,$$

where l_2 is a mean free path for momentum transfer between protons *behind* the shock. Note that $l_2/l \sim \tau_2^2 \sim M^4$, so that L may be very large for strong shocks.

Peak proton temperature

A crude estimate of the peak proton temperature, $\theta_{\max}$, before protons and electrons attain equilibrium at T_2 can be made by neglecting all heat interchange between protons and electrons, i.e. putting $\tau = \tau_2$ and supposing too that θ reaches $\theta_{\max}$ as ν reaches $\nu_{\max} \doteqdot 4$. The computed solutions show that these assumptions are reasonable for the estimation of $\theta_{\max}$. From (3.21) and (3.23), it is found that

$$\theta_{\max} \doteqdot 1{\cdot}28\tau_2.$$

This is an overestimate as the heat interchange from protons to electrons is neglected and θ reaches $\theta_{\max}$ before ν reaches $\nu_{\max}$.

The electric field

An electric field is caused by polarization imposed by the boundary conditions. In a plasma an electric field, together with a density and thermal gradient of electrons, gives rise to a net electron drift or current relative to the ions. Spitzer (1956) has given the relation between them in terms of a coefficient of electrical conductivity σ and other coefficients which he has calculated. Evaluating these coefficients the expression for the electron current relative to the ions at rest is

$$j = \sigma\left[E + \frac{kT_e}{en_e}\frac{dn_e}{dx} + 1{\cdot}70\frac{k}{e}\frac{dT_e}{dx}\right]. \tag{5.2}$$

In a steady state the boundary conditions prohibit any total current and, therefore, $j = 0$ (neglecting any convected charge). From (5.2)

$$E = -\frac{dV}{dx} = -\frac{kT_e}{e}\frac{1}{l}\frac{d}{d\xi}[\log\nu + 1{\cdot}70\log\tau]. \tag{5.3}$$

Let $Q \equiv eV/kT_1$, V being the potential, so that

$$\frac{dQ}{d\xi} = \tau\frac{d}{d\xi}[\log\nu + 1{\cdot}70\log\tau]. \tag{5.4}$$

If $\tau \doteqdot \tau_2$, equation (5.4) may be integrated by parts to give

$$Q - Q_1 = \tau_2 \log \nu + 1{\cdot}70(\tau - 1). \tag{5.5}$$

In particular,

$$Q_2 - Q_1 = \tau_2(\log 4 + 1{\cdot}70) = 3{\cdot}1\tau_2.$$

Equation (5.3) gives an estimate for E, so we are in a position to justify the neglect of the electrostatic force in (3.4) and the assumption that $n_i \doteqdot n_e$. Since $E \sim (kT)/(el)$, the ratio of the electrostatic 'pressure' $E^2/(8\pi)$ to the gas pressure p can be written

$$\frac{E^2}{8\pi p} \sim \left(\frac{d}{l}\right)^2,$$

where d is the Debye length defined as $d \equiv (kT/4\pi e^2 n)^{1/2}$.

Also, using $\operatorname{div} E = 4\pi e(n_i - n_e)$,

$$\frac{n_i - n_e}{n} \sim \left(\frac{d}{l}\right)^2.$$

Over an enormous range of conditions $l \gg d$, thus justifying the assumptions.

6. Conclusions

For a strong shock wave ($M > 2$), a quasi-equilibrium theory which assumes the protons and electrons to have separate equilibrium temperatures leads to the following results.

(*a*) Density and velocity and the proton temperature all change in one or two mean free paths (l_2) for momentum transfer in the hot (shocked) gas.

(*b*) The electron temperature changes more gradually over a larger distance ($\sim l_2/2P \doteqdot 8l_2$). This is equivalent to the mean free path for energy transfer between electrons and protons.

(*c*) The proton temperature rises to a maximum, which is only slightly higher than the final equilibrium temperature T_2.

(*d*) Density and total pressure increase monotonically through the shock.

(*e*) An electric field is set up across the shock caused by the density and temperature gradients of the electrons.

A weak shock ($M \doteqdot 1$) is broad ($L \sim l_2/\{4P(M-1)\}$) and in it electrons and protons are always in thermal equilibrium.

The author wishes to thank Mr R. Pease and Dr W. Marshall for their helpful reading of this paper.

References

Becker, R. 1923 *Z. Phys.* **8**, 321.

Chapman, S. & Cowling, T. G. 1939 *Mathematical Theory of Non-Uniform Gases.* Cambridge University Press.

Mott-Smith, H. M. 1951 *Phys. Rev.* **82**, 885.

Post, R. 1956 *Rev. Mod. Phys.* **28**, 338.

Spitzer, L. 1956 *Physics of Fully Ionized Gases.* New York : Interscience.

Thomas, L. 1944 *J. Chem. Phys.* **12**, 449.